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COMMENT

Critical exponent of a directed self-avoiding walk

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Abstract. It has been shown that the angular correlation range is infinite for the directed self-avoiding walk problem on the square lattice. This result implies a one-dimensional-like critical exponent, namely $\nu = 1$.

The directed self-avoiding random walk problem (DSAW) on the square lattice has been recently investigated numerically by Chakrabarti and Manna (1983). In this SAW model, steps in the x direction are restricted to be only in the positive x direction. Chakrabarti and Manna calculated numerically the average end-to-end distance $\langle R_n \rangle$ for walks of n steps, with values of n up to 14. They found that for large n

$$\langle R_n \rangle \simeq n^{\nu} \tag{1}$$

with $\nu = 0.86 \pm 0.02$. Redner and Majid (1983) using the transfer matrix approach, Szpilka (1983) using the method of generating functions, Cardy (1983) from a field theoretic approach and Blöte and Hilhorst (1983) using combinatorial arguments all disproved the Chakrabarti and Manna claim and proved that $\nu = 1$. In addition, Redner and Majid calculated the following asymptotic expressions for the persistence length along the x axis and the mean square displacement respectively

$$\langle x_n \rangle = \tfrac{1}{2} n + \tfrac{1}{2}(\sqrt{2} - 1) \tag{2a}$$

$$\langle R_n^2 \rangle = \tfrac{1}{4} n^2 + \tfrac{7}{8}\sqrt{2}\, n + \tfrac{1}{4}(\tfrac{17}{2}\sqrt{2} - 1). \tag{2b}$$

Szpilka obtained the same results up to order n.

In this comment we utilise the Markovian nature of the problem to obtain closed expressions for the angular correlation functions and the radial moments that are derived from them. Let θ_{ij} denote the angle between the directions of the ith and jth steps. The average $\langle \cos(\theta_{ij}) \rangle_n$, $(i \leq j)$, contains two types of contributions: (i) all walks with i and j parallel to the x axis; and (ii) all walks with i, j and k pointing in the $+y$ direction for all k obeying $i < k < j$ and with their reflections about the x axis. The contribution of any walk with i and j pointing in the $+y$ direction and at least one kink at the kth step $(i < k < j)$ is averaged out by the contribution of the walk obtained from it by a reflection of the $n-k$ part of the walk about the kink. Thus

$$\langle \cos(\theta_{ij}) \rangle_n = (1/c_n)(c_i(x) c_{j-1}(x) c_{n-j} + 2 c_i(y) \tilde{c}_{n-j}) \tag{3}$$

where c_n is the total number of DSAW with n steps, $c_n(x)$ and $c_n(y)$ are the number of n-steps DSAW with the last step pointing in the x or $+y$ direction respectively and

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$\tilde{c}_n$ is the number of n-step DSAW with the first step not allowed to point in the $-y$ direction. Fisher and Sykes (1959), while deriving lower bounds for the connective constant for SAW on the square lattice, calculated c_n:

$$c_n = \tfrac{1}{2}(\lambda_+^{n+1} + \lambda_-^{n+1}) \tag{4}$$

where $\lambda_\pm = 1 \pm \sqrt{2}$.

It is easy to see that $c_n(x) = c_{n-1}$, $c_n(y) = c_n(-y) = \tfrac{1}{2}(c_n - c_{n-1})$ and $\tilde{c}_n = \tfrac{1}{2}(c_{n+1} - c_n)$ (it should be noted that $c_{-1} = 1$). Equation (3) becomes

$$\begin{aligned}\langle\cos(\theta_{ij})\rangle_n &= \frac{1}{c_n}[c_{i-1}c_{j-i-1}c_{n-j} + \tfrac{1}{2}(c_i - c_{i-1})(c_{n+1-j} - c_{n-j})] \\ &= \frac{1}{4[1+(\lambda_-/\lambda_+)^{n+1}]}\{[1+(\lambda_-/\lambda_+)^i][1+(\lambda_-/\lambda_+)^{j-i}][1+(\lambda_-/\lambda_+)^{n+1-j}] \\ &\quad + 2\lambda_+^{i-j}[1-(\lambda_-/\lambda_+)^i][1-(\lambda_-/\lambda_+)^{n+1-j}]\}.\end{aligned} \tag{5}$$

Obviously for large $j-i$ values the first term on the RHS of equation (5) representing the type a contribution is much larger than the second term representing the type b contribution to the angular average. For large n and i, j well inside the chain, equation (5) reduces to

$$\langle\cos(\theta_{ij})\rangle_n = \tfrac{1}{4}[1+(\lambda_-/\lambda_+)^{j-i} + 2\lambda_+^{i-j}]. \tag{6}$$

The angular average does not decay to zero for large separations $j-i$, but converges to a constant value. This infinite range of angular correlation reflects the exclusion of reversals in the x direction in the model, and implies a one-dimensional-like critical exponent $\nu = 1$. The persistence length $\langle x_n\rangle$ is given by the sum of the projections on the x axis

$$\langle x_n\rangle = \frac{1}{c_n}\sum_{j=1}^{n} c_{j-1}c_{n-j} = \tfrac{1}{2}n - \tfrac{1}{2} + \tfrac{1}{4}\sqrt{2} + O(\lambda_-/\lambda_+)^n. \tag{7}$$

The even moments of the end-to-end distance distribution function are given by averages of powers of the cosines (Baram and Gelbart 1977, Baram 1982)

$$\langle R_n^{2k}\rangle = \left\langle\left(n + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\cos(\theta_{ij})\right)^k\right\rangle_n. \tag{8}$$

In particular the second moment is given by

$$\langle R_n^2\rangle = n + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\langle\cos(\theta_{ij})\rangle = \tfrac{1}{4}n^2 + \tfrac{7}{8}\sqrt{2}\, n - \tfrac{1}{4}(\tfrac{3}{2}\sqrt{2}+1) + O(\lambda_-/\lambda_+)^n. \tag{9}$$

The constant terms in equations (7) and (9) disagree with the constant terms of Redner and Majid, equations (2a) and (2b). Our results for $\langle x_n\rangle$ and $\langle R_n^2\rangle$ agree with enumeration results to five decimal places by order seven and nine respectively.

The leading term in the expansion of the fourth moment is $\frac{1}{16}n^4$. Therefore the ratio $\langle R_n^4\rangle/\langle R_n^2\rangle^2$ tends asymptotically to unity, indicating that the distribution function is highly concentrated at the point $x = n/2$, $y = 0$.

References

Baram A 1982 *J. Chem. Phys.* **77** 5255
Baram A and Gelbart W M 1977 *J. Chem. Phys.* **66** 4666
Blöte H W J and Hilhorst H J 1983 *J. Phys. A: Math. Gen.* **16** 3687
Cardy J L 1983 *J. Phys. A: Math. Gen.* **16** L355
Chakrabarti B K and Manna S S 1983 *J. Phys. A: Math. Gen.* **16** L113
Fisher M E and Sykes M F 1959 *Phys. Rev.* **114** 45
Redner S and Majid I 1983 *J. Phys. A: Math. Gen.* **16** L307
Szpilka A M 1983 *J. Phys. A: Math. Gen.* **16** 2883